LECTURE

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THE REAL NUMBERS: SOME BASIC CONCEPTS

The set of real numbers, denoted by R, is a totally ordered field

(R,+,·,≥) meaning that

• (R,+,·) is a field, where 0 and 1 are the neutral elements of + and ·, respectively (notice that 0 = 1);

• ≥ is an order relation on R, i.e., a binary relation, which is reflexive, transitive and antisym- metric;

• ≥ is total, i.e., ∀x, y ∈ R we have x ≥ y or y ≥ x;

• ≥ is compatible with +, i.e., ∀x,y,z ∈ R we have x + z ≥ y + z whenever x ≥ y;

• ≥ is compatible with ·, i.e., ∀x, y ∈ R s.t. x ≥ 0 and y ≥ 0, we have xy ≥ 0. As usual, we associate to ≥ the inverse order relation ≤ as well as the strict order relations > and <, defined for any x, y ∈ R by

x ≤ y ⇔ y ≥ x; x>y ⇔ x ≥ y and x = y; x<y ⇔ y > x.

Proposition 1.1 We have x2 ≥ 0 for all x ∈ R. Consequently, 1 > 0.

Definition 1.2 For any subset A of R we introduce the following (possibly empty!) sets

lb(A) . .

= {x ∈ R | x ≤ a, ∀a ∈ A}; ub(A) . .

= {x ∈ R | x ≥ a, ∀a ∈ A}.

A number x ∈ R is said to be a

• lower bound of A if x ∈ lb(A);

• upper bound of A if x ∈ ub(A);

• least element (or minimum) of A if x ∈ A ∩ lb(A);

• greatest element (or maximum) of A if x ∈ A ∩ ub(A).

Remark 1.3 Every set A ⊆ R has at most one least element and, if it exists, we denote it by minA. Similarly, A has at most one greatest element and, if it exists, we denote it by maxA.

Definition 1.4 A subset A of R is said to be

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• bounded (from) below, if A has lower bounds, i.e., lb(A) = ∅;

• bounded (from) above, if A has upper bounds, i.e., ub(A) = ∅;

• bounded, if A is both bounded above and below;

• unbounded, if A is not bounded.

Remark 1.5 The empty set is bounded. More precisely, we have

lb(∅) = ub(∅) = R.

Example 1.6 (i) A = {a ∈ R | a ≥ 2}: unbounded (since it is not bounded above), bounded below by any v ≤ 2, minA = 2. (ii) A = {a ∈ R | 0 <a< 1}: bounded (above by any u ≥ 1, below by any v ≤ 0), no minimum, no maximum. (iii) A =

{

1 n

} | n ∈ N∗

: bounded (above by any u ≥ 1, below by any v ≤ 0), maxA = 1, no

minimum. (iv) Every nonempty finite set has a minimum and a maximum.

Proposition 1.7 (Completeness Axiom) The totally ordered field of real numbers (R,+,·,≥) is complete, meaning that every nonempty set A ⊆ R that is bounded above has a least upper bound, denoted by supA and called the supremum of A. In other words, we have

supA . .

= min(ub(A)).

Alternatively, every nonempty set A ⊆ R that is bounded below has a greatest lower bound, denoted by inf A and called the infimum of A. In other words,

inf A . .

= max(lb(A)).

Example 1.8 (i) A = {a ∈ Z | −3 2

√

2}: maxA = supA = 1, minA = inf A = −1. (ii); A = {a ∈ R | 0 < a ≤ 1}: maxA = supA = 1, inf A = 0, no minimum.

Remark 1.9 The Completeness Axiom is also known in the literature as the Supremum Property, since it shows that every nonempty subset of R which is bounded above has a supremum in R. Its counterpart shows that every nonempty subset of R which is bounded below has an infimum in R. Indeed, let A ⊆ R, A = ∅, bounded below. Then the set −A = {−a | a ∈ A} is nonempty and bounded above, so, by the Supremum Property, it has a supremum in R. Thus we have inf A =−sup(−A).

Remark 1.10 Let A ⊆ R be a nonempty set. If A has a greatest element (resp. a least element), then supA = maxA (resp. inf A = minA). Conversely, if A is bounded above and supA ∈ A (resp. A is bounded below and inf A ∈ A), then supA = maxA (resp. inf A = minA).

Definition 1.11 We attach to R two elements −∞ and +∞ (or ∞) s.t.

∀x ∈ R, −∞ < x and x < +∞.

The set R . .

≤ a ≤

= R ∪ {−∞,+∞} is called the extended real number system. If a set A ⊆ R is not bounded above, we define supA . .

= +∞. If a set A ⊆ R is not bounded below, we define inf A . .

= −∞. Also, we define sup∅ . .

= −∞ and inf ∅ . .

= +∞ (see Remark 1.5!).

We denote by N . .

= {1,2 . .

=1+1,3 . .

=1+1+1,...} the set of natural numbers.

Remark 1.12 N is the smallest inductive subset of R w.r.t. inclusion (a set A ⊆ R is said to be inductive if 1 ∈ A and x + 1 ∈ A whenever x ∈ A). We have minN = 1 and for every n ∈ N, n<n + 1 and {x ∈ N | n<x<n + 1} = ∅. Every nonempty subset of N has a least element.

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∈ Proposition 1.13 (Principle of Mathematical Induction) Let n

0

N and let P(n) be a prop- erty defined for any number n ∈ N, n ≥ n

0

. Suppose that the following two conditions hold: I. P(n

0

) is true; II. If P(k) is true for some k ∈ N, k ≥ n

0

, then P(k + 1) is also true. Then we have

III. P(n) is true, ∀n ∈ N, n ≥ n

0

.

The following result is a consequence of the Completeness Axiom (Supremum Property).

Corollary 1.14 (Archimedean Property) The set of natural numbers N is not bounded from above. In other words, for every x ∈ R there exists n ∈ N s.t. n>x.

Proof. Suppose x ≥ n, ∀n ∈ N. Then N is nonempty and bounded above by x, so, by Theorem 1.7, it has a supremum u ∈ R. Since u − 1 < u, u − 1 cannot be an upper bound of N. This means that ∃m ∈ N s.t. u − 1 < m. Thus, u<m + 1 ∈ N, which is a contradiction to the fact that u is an upper bound of N. D

The sets of integer numbers and rational numbers are defined as

Z . .

= {m − n | m, n ∈ N}; Q . .

= {mn−1 | m ∈ Z,n ∈ N}.

Remarks 1.15 1. For every x ∈ R there is a unique k ∈ Z such that k ≤ x<k +1; we denote this k by [x] or ⌊x⌋ and call it the integer part or floor of x. x = 2. 3. yn We For (when have

every n √

≥ n 2 2 /∈ ∈ we Q. N denote and Therefore x y ∈ = R, the √

n

x ≥ 0, there exists a unique number y ∈ R, y ≥ 0 such that x).

set R \ Q of the so-called irrational numbers is nonempty.

As a consequence of the Archimedean Property we obtain the following result:

Corollary 1.16 (Density of Q in R) For any real numbers a, b ∈ R such that a<b there exists x ∈ Q such that a<x<b.

Proof. Let a, b ∈ R such that a<b. By the Archimedean Property (Corollary 1.14) we can find a number n ∈ N s.t. n > 1

b−a

, i.e.,

nb − 1 > na (1.1)

Case 1: in demand.

If nb ∈ Z then (1.1) shows that a < nb−1

n

< b, hence x . .

= nb−1

n

∈ Q satisfies the property

Case 2: If nb /∈ Z then we consider the integer part of nb, namely m . .

= [nb]. In this case we have

m<nb<m + 1. (1.2)

By (1.1) and (1.2) we deduce that m > nb − 1 > na hence na < m < nb. Thus, in this case the number x . .

= m n

∈ Q satisfies a<x<b. D

Remark 1.17 (Q,+,·,≥) is a totally ordered field but, in contrast to (R,+,·,≥), it does not satisfy the Completeness Axiom. However, for every x ∈ R we have

sup{y ∈ Q | y<x} = x = inf{y ∈ Q | y>x}; sup{z ∈ R \ Q | z<x} = x = inf{z ∈ R \ Q | z>x}.

Next we present some properties which are of practical interest.

Proposition 1.18 If A ⊆ B ⊆ R are nonempty bounded sets, then

inf B ≤ inf A ≤ supA ≤ supB.

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Proposition 1.19 If A and B are nonempty subsets of R which are bounded above, then A ∪ B is bounded above and the following relations hold:

sup(A ∪ B) = max{supA,supB};

inf(A ∪ B) = min{inf A,inf B};

Proposition 1.20 For any nonempty subsets A and B of R, we have

sup(A + B) = supA + supB, inf(A + B) = inf A + inf B,

where A + B . .

= {a + b | a ∈ A, b ∈ B}.

If f : D → R is a function, defined on a nonempty set D, then it will be convenient to denote

x∈D inf

f(x) . .

= inf f(D) and sup x∈D

f(x) . .

= supf(D),

where f(D) = Im(f) . .

= {f(x) | x ∈ D} represents the function’s image. In particular, if D = N, a function f : N → R represents a sequence (x

n

)

n∈N

. In this case we will write

n∈N inf

x

n

. .

= inf{x

n

| n ∈ N} and sup

x

n

. .

= sup{x

n

| n ∈ N}. n∈N

The following result is another important consequence of the Completeness Axiom (Supremum Prop- erty).

Corollary 1.21 (Nested Interval Property) Consider a sequence of closed intervals I

n= [an,bn]⊆ R, with an< bn for all n ∈ N. If In⊇ In+1 for all n ∈ N, i.e.,I1 ⊇ I2⊇ ... ⊇ In ⊇ In+1⊇ ... is a nested sequence of closed intervals, then we have ∞⋂ n=1 In = ∅ (i.e., ∃x ∈ R s.t. ∀n ∈ N,x ∈ In). Proof. Let A = {ak | k ∈ N}. Then, ∀n ∈ N, bn is an upper bound of A. Hence A is nonempty and bounded above. By the Completeness Axiom (Proposition 1.7), we deduce that A has a supremum in R. Thus, ∀n ∈ N, an ≤ supA ≤ bn. This shows that supA ∈∞⋂ In . Dn=1

Definition 1.22 A set V ⊆ R is said to be

• a neighborhood of a number x ∈ R, if there exists a real number ε > 0 such that (x−ε, x+ε) ⊆ V ;

• a neighborhood of −∞, if there exists a number a ∈ R such that (−∞,a) ⊆ V ;

• a neighborhood of +∞, if there exists a number a ∈ R such that (a,+∞) ⊆ V .

Proposition 1.23 Let x ∈ R. Then

(i) if x ∈ R and V ∈ V(x), then x ∈ V . (ii) if V ∈ V(x) and U ⊆ R s.t. V ⊆ U, then U ∈ V(x). (iii) if U, V ∈ V(x), then U ∩ V ∈ V(x).

Theorem 1.24 Let A ⊆ R be a nonempty set, which is bounded from below by α ∈ R. Then the following assertions are equivalent:

1◦ inf A = α. 2◦ For every real number β>α there exists x ∈ A such that x<β. 3◦ For every real number ε > 0 we have A ∩ [α, α + ε) = ∅. 4◦ For every V ∈ V(α) we have V ∩ A = ∅.

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Corollary 1.25 Let A ⊆ R be a nonempty set, which is bounded from above by α ∈ R. Then the following assertions are equivalent:

1◦ supA = α. 2◦ For every real number β<α there exists x ∈ A such that x>β. 3◦ For every real number ε > 0 we have A ∩ (α − ε, α] = ∅. 4◦ For every V ∈ V(α) we have V ∩ A = ∅.

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